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Sufficient conditions for a group of automorphisms of a Riemann surface to be its full automorphism group

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Dedicated to the memory of Sheela Phansalkar (1966–1990)

Abstract

There are few examples in the literature of Riemann surfaces whose defining algebraic equations and full automorphism groups are completely determined. Although explicit examples of Riemann surfaces which admit automorphisms may be constructed by the use of symmetries in the defining equations of the surface, determining whether the admitted automorphisms constitute the full automorphism group is usually intractable. In this paper, it is proved that for many groups a simple lifting criterion determines whether the admitted automorphisms form the full automorphism group. The criterion is employed to give numerous examples of Riemann surfaces whose defining equations and full automorphism groups are determined. © 1998 Elsevier Science B.V.

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0. Introduction

There is a large literature concerning Riemann surfaces which admit nontrivial automorphisms, in particular, an emphasis has been placed on finding groups of automorphisms which are maximal in some sense. For example, let X be a compact Riemann surface of genus $g_X \geq 2$ with automorphism group $Aut(X)$. It is well known that X can have at most $84(g_X - 1)$ automorphisms. Groups isomorphic to the automorphism group of a Riemann surface which possesses this maximal number of automorphisms

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are called Hurwitz groups and an extensive amount of research has been done to determine which groups are Hurwitz groups. Another example is the specification of a certain class of groups, say cyclic groups or nilpotent groups, and the determination of the maximal order of any group H of automorphisms in this class which a Riemann surface X may admit [4, 10]. Often in these cases however, the question of whether the admitted automorphism group is, in fact, the full automorphism group of the Riemann surface is not addressed. In addition, a Riemann surface which arises in such a manner often has the Riemann sphere as the orbit space under its automorphism group. As a result, little attention has been paid to Riemann surfaces which have orbit spaces of genus greater than zero under their automorphism groups.

Again let X be a Riemann surface of genus $g_X \geq 2$. If the orbit space of X under $Aut(X)$ has genus g, we say X has a genus g quotient. In this paper we examine Riemann surfaces which have a genus g quotient, for $g \geq 1$. The fundamental problem considered is the following: If X admits a group of automorphisms H, and if the orbit space X/H has genus $g \geq 1$, under what conditions can we be assured that $H = Aut(X)$, the full automorphism group of X. Although this is a difficult question in general, in this paper a criterion is given which can be applied to a large number of automorphism groups to determine which are the full automorphism groups of a given Riemann surface. Techniques involving Fuchsian groups are employed in the proofs of theorems of this paper; however the statements of the theorems are independent of Fuchsian groups. This has an important benefit concerning the construction of explicit examples of Riemann surfaces whose defining equations and full automorphism groups are completely determined. We describe this below.

There are two main ways to consider a compact Riemann surface, either as the upper half plane U under the action of a Fuchsian group, or as the nonsingular model for a function field of transcendence degree one over $\mathbb{C}$, as is done in algebraic geometry. The strengths of each method, unfortunately, do not easily allow the construction of explicit examples of Riemann surfaces with desired automorphism groups. For example, techniques involving Fuchsian groups can be used to show the existence of large numbers of Riemann surfaces whose full automorphism groups are completely determined. In general, they are obtained by mapping an appropriate Fuchsian group Λ onto a finite group H. If Γ denotes the kernel of the homomorphism, then the Riemann surface $X = U/\Gamma$ admits H as a group of automorphisms. If Λ is chosen, for example, to be a finitely maximal Fuchsian group, and if Γ contains no nonidentity elements of finite order, then $H = Aut(X)$. Unfortunately, although this method gives a large amount of information about Riemann surfaces which admit automorphisms, it rarely yields defining algebraic equations for the Riemann surfaces found. In fact, given the defining equations of a Riemann surface, I am unaware of a method which determines whether a Fuchsian group corresponding to the surface is finitely maximal or not.

On the other hand, using techniques of classical algebraic geometry it is easy to give the defining algebraic equations of a Riemann surface X which admits nontrivial automorphisms by using symmetries in the defining equations for X. However, it is rarely possible to directly determine if the admitted automorphisms constitute the full automorphism group of the Riemann surface. The theorems of this paper can be applied to a Riemann surface defined explicitly by a set of algebraic equations. The results of this paper, in conjunction with families of Riemann surfaces which have a trivial automorphism group, yield a limitless number of examples of Riemann surfaces whose defining equations and full automorphism groups are completely determined. Examples are considered in Section 4.

Our main results are Theorems 2.4, 2.8, 3.1, and Corollary 3.5. Each theorem has a similar flavor. Let X be a compact Riemann surface of genus greater than one, let $H \leq Aut(X)$, let $Y = X/H$, and let $g \geq 1$ denote the genus of Y. If H satisfies certain hypotheses, then $H = Aut(X)$ if and only if no nonidentity automorphism of Y lifts to X. If the automorphism group of Y is completely determined, it can, at least in theory, be ascertained if $H = Aut(X)$. In particular, if $Aut(Y)$ is trivial, then $H = Aut(X)$.

The paper is organized as follows. In Section 1 we have preliminary remarks concerning Fuchsian groups. In Section 2 we prove a theorem concerning simple groups, and groups whose orders are not divisible by “small” primes. In Section 3 we consider nilpotent, fixed-point free automorphism groups. In Section 4 we apply our results to give explicit equations of Riemann surfaces whose defining equations and full automorphism groups are completely determined. In Section 5 we restrict our attention to simple groups of automorphisms which yield a genus 3 orbit space.

1. Preliminaries

All Riemann surfaces are assumed to be compact. For basic facts concerning Fuchsian groups and automorphism groups arising from mapping a Fuchsian group onto a finite group see [8]. We emphasize here only a few relevant points.

Let Δ be a Fuchsian group of signature $(g; e_1, \ldots, e_r)$ with $g \geq 1$ and let the finite group H be a homomorphic image of Δ. Let U denote the upper half plane and let Γ denote the kernel of this homomorphism. Then the Riemann surface $X = U/\Gamma$ admits H as a group of automorphisms, and $Y = U/\Delta$ is the orbit space of X under H. In addition, if Γ contains no elliptic elements and if $g \geq 2$, then the full automorphism group of X is $N(\Gamma)/\Gamma$, where $N(\Gamma)$ denotes the normalizer of Γ in $PSL(2,\mathbb{R})$. If Δ is chosen to be finitely maximal, in other words, not contained as a subgroup of finite index in another Fuchsian group, then obviously Δ is self normalizing in $PSL(2,\mathbb{R})$, and thus $H = Aut(X)$.

Unfortunately, although “most” Fuchsian groups of signature $(g; e_1, \ldots, e_r)$ with $g \geq 2$ are finitely maximal, it is practically impossible to tell if a Riemann surface defined by a set of algebraic equations has an associated Fuchsian group which is finitely maximal or not. The problem is that knowledge of the Fuchsian group associated to a Riemann surface rarely yields information about its defining equations. Thus, if we search for an explicit example of a Riemann surface X which has a particular full automorphism group, $H = Aut(X)$, then it is desirable to have a criterion independent of Fuchsian groups to determine if a Riemann surface which admits H as a group of automorphisms has H as its full automorphism group. What the main theorems of this paper yield is that for many groups it can be determined if H is the full automorphism group of X by examining the automorphism group of X/H.

Let X be a compact Riemann surface of genus g_x which admits a group of automorphisms H, let Y be the orbit space of X under H, and let g denote the genus of Y. Assume that branching occurs at r points of Y, say $P_1, \ldots, P_r$, and let the branching index at P_i be $m_i - 1$ (in other words at these points the map from X to Y is m_i to one and the ramification index at the point P_i is m_i). We recall the Riemann Hurwitz formula [2]

$$2(g_x - 1) = |H| \left(2(g-1) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{1}$$

We note that this formula is true even if Y does not arise as an orbit space, but is merely a Riemann surface which X covers as long as the following condition is satisfied: If P is a point of Y at which branching occurs, then all points of X which lie over P must have the same branch index. In this case, $|H|$ in (1) is replaced by the degree of the map from X to Y.

Definition 1.1. Let X be a Riemann surface of genus greater than one and let $G = Aut(X)$. If X/G has genus g we say X has a genus g quotient.

Proposition 1.2. *Let X be a Riemann surface, $H \leq Aut(X)$ and let X/H have genus $g \geq 2$. Then $|H| \leq (g_x - 1)/(g - 1)$. Equality holds if and only if X is unbranched over its orbit space under H.*

Proof. Immediate from (1). □

Definition 1.3. Let X be a Riemann surface, let $H \leq Aut(X)$, and let $Y = X/H$. Let $\sigma \in Aut(Y)$ and let $\pi : X \to Y$ be the map which takes a point of X to its orbit under H. We say σ lifts to $\sigma' \in Aut(X)$ if $\pi\sigma' = \sigma\pi$.

There is a natural setting in which $\sigma \in Aut(Y)$ lifts to $\sigma' \in Aut(X)$. Let Λ be a Fuchsian group and suppose $\eta \in PSL(2, \mathbb{R})$ normalizes Λ. Then $\eta\Lambda$ is an automorphism of $Y = U/\Lambda$. If $\Gamma \lhd \Lambda$ and η normalizes Γ, then $\eta\Lambda$ lifts to the automorphism $\eta\Gamma$ of $X = U/\Gamma$.

2. A property concerning normalizers

We begin with a group theory definition.

Definition 2.1. Let H be an arbitrary finite group and let k be a fixed positive integer. We say H has the "normalizer property for index k," denoted by $N(k)$, if whenever H

is a subgroup of a group G with $[G:H] \leq k$ then $H < N_G(H)$ with strict inequality holding.

Remark. Let $(g; e_1, \dots, e_i, \dots, e_n)$, with $n \geq 0$, be a sequence of positive integers with $g \geq 1$ and each $e_i \geq 2$. We associate an integer k to the sequence as follows. If each $e_i = 2$ or 3, or if the sequence is (g), we associate the integer $42(2g+n-2)$ to it. If each e_i is 2,4 or 5, with a 5 actually appearing, let $k = 20(2g+n-2)$. If each e_i is a 2,4 or 7, with a 4 and 7 actually appearing, let $k = 28(2g+n-2)/3$. In all other cases, let L equal the least common multiple of all the e_i's greater than 3. If $L = 4,5$ or 6, define $k = 6L(2g+n-2)/(L-3)$. If $L \geq 7$, define $k = 6L(2g+n-2)/(L-6)$.

Proposition 2.2. *Let X be a compact Riemann surface of genus greater than one, let $H \leq Aut(X)$, let $Y = X/H$, and let $g \geq 1$ denote the genus of Y. Assume exactly n points of Y are ramified in X and let $e_1, \dots, e_n$ be the ramification indices at these points. Let k be calculated, as in the Remark, for $(g; e_1, \dots, e_n)$. Assume H has the property $N(k)$. Then $H \neq Aut(X)$ if and only if there is a nonidentity automorphism of Y which lifts to an automorphism of X.*

Proof. Assume H is not the full automorphism group of X. Let Γ be a fixed point free Fuchsian group such that U/Γ is biholomorphic to X. Since $H \leq Aut(X)$, there exists a Fuchsian group Δ such that $\Gamma \lhd \Delta$ and $\Delta/\Gamma \cong H$. We identify Y with U/Δ and note that Δ has signature $(g; e_1, \dots, e_n)$. Let $\Lambda = N(\Gamma)$; recall that $G = N(\Gamma)/\Gamma \cong Aut(X)$. Assume the signature of Λ is $(g_0; f_1, \dots, f_m)$. Then

$$|G|\left((2g_0-2)+\sum_{i=1}^{m}(1-1/f_i)\right) = 2g_X - 2 = |H|\left((2g-2)+\sum_{i=1}^{n}(1-1/e_i)\right). \tag{2}$$

Thus,

$$[G:H] = \frac{((2g-2)+\sum_{i=1}^{n}(1-1/e_i))}{((2g_0-2)+\sum_{i=1}^{m}(1-1/f_i))}. \tag{3}$$

The numerator of (3) is clearly less than or equal to $2g+n-2$. Note that each e_i must divide one of the f_j's. The denominator will have minimal size if the signature of Λ is $(0; f_1, f_2, f_3)$. One quickly concludes that if $e_i = 2,4$ or 5 for each i with $1 \leq i \leq n$, then the minimal value of the denominator occurs if the signature of Λ is $(0; 2,4,5)$. Similarly, if each e_i is 2,4 or 7, then the minimal value of the denominator occurs if the signature of Λ is $(2,4,7)$. If $L = 4,5$, or 6, then the minimal value for the denominator occurs if the signature of Λ is $(0; 2,3,2L)$. Similarly, if $L \geq 7$, then the minimal value for the denominator occurs if the signature of Λ is $(0; 2,3,L)$. For the remaining cases, the minimal value of the denominator occurs when the signature of Λ is $(0; 2,3,7)$. Thus $[G:H] \leq k$. Thus, if H is not the full automorphism group of X, then $1 < [\Lambda : \Delta] = [G:H] \leq k$. Since H has the property $N(k)$, there exists an element $\eta \in \Lambda$, with $\eta \notin \Delta$, such that η normalizes Δ. Thus, $\eta\Delta$ induces a nontrivial automorphism of $Y = U/\Delta$ and the automorphism $\eta\Delta$ of Y lifts to the automorphism $\eta\Gamma$ of X. Thus, there exists a nonidentity automorphism of Y which lifts to X. The converse statement is trivial. □

From Proposition 2.2, whether a group which satisfies $N(k)$ for suitable k is the full automorphism group of a Riemann surface depends only on the automorphism group of the orbit space. We now determine two large classes of groups which satisfy $N(k)$ for suitable k. These are purely group theoretic results; however our goal is to specialize to the case of automorphisms of Riemann surfaces.

Proposition 2.3. *Let H be an arbitrary finite group and $k' > 1$ an arbitrary integer. Assume there exists a finite group G with $H < G$, $[G : H] = k'$, and $N_G(H) = H$. Suppose the maximal subgroups of H have $m_1, \ldots, m_s$ as their respective indices in H. Then there exist nonnegative integers $t_1, \ldots, t_s$ such that*

$$k' = [G : H] = 1 + \sum_{i=1}^{s} t_i m_i. \tag{4}$$

Proof. Write G as a disjoint union of double cosets: $G = Hg_1H \cup Hg_2H \cup \cdots \cup Hg_rH$. We may assume $g_1 \in H$. It is well known that the number of cosets of H contained in Hg_jH equals $[H : H^{g_j} \cap H]$. Since $H = N_G(H)$, we have $[H : H^{g_j} \cap H] > 1$ for $j > 1$, in fact, it must be a multiple of one of the m_i. Since $Hg_1H = H$ we obtain (4). □

Theorem 2.4. *Let X be a Riemann surface of genus greater than one, let $H \leq Aut(X)$, let $Y = X/H$, and let $g \geq 1$ denote the genus of Y. Assume n points of Y are ramified in X and let k be calculated as in the Remark. In addition, assume H contains no proper subgroup of index less than k. Then $H = Aut(X)$ if and only if no nonidentity automorphism of Y lifts to X. In particular, if the order of H is not divisible by a prime less than k, then $H = Aut(X)$ if and only if no nonidentity automorphism of Y lifts to X.*

Proof. Proposition 2.3 and the hypothesis on H guarantee that H satisfies $N(k)$. Proposition 2.2 yields the conclusion. □

Corollary 2.5. *Let X be a Riemann surface of genus $g_X \geq 2$, let $H \leq Aut(X)$ have order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$, and let $Y = X/H$ have genus $g \geq 1$. Assume at most n points of Y are ramified in X, and assume that for each i, with $1 \leq i \leq s$, there is an element of H of order p_i which is not fixed point free on X. Furthermore, assume each prime satisfies $p_i \geq 6(2g+n-1)$. Then $H = Aut(X)$ if and only if no nonidentity automorphism of Y lifts to X.*

Proof. Fix i, for some $i \leq s$, and let $p = p_i$. By assumption, there is an element of H of order p which is not fixed point free on X. Thus, the ramification index at some point of Y must be divisible by p^{β} for some β with $1 \le \beta \le \alpha_i$. Thus, if k is calculated as in the Remark, p^{β} divides L. But the function $L/(L-6)$ is a decreasing function if $L > 6$, thus $p/(p-6) \ge p^{\beta}/(p^{\beta}-6) \ge L/(L-6)$. In addition, $p \ge 6p(2g+n-2)/(p-6)$ if and only if $p \ge 6(2g+n-1)$. Thus if $p \ge 6(2g+n-1)$, then $p \ge 6p(2g+n-2)/(p-6) \ge 6L(2g+n-2)/(L-6)$. Thus, each prime which divides the order of H is greater than or equal to k. Thus, H satisfies $N(k)$. The result follows from Theorem 2.4. □

The above corollary is of great utility if only a bound on the number of points ramified is known, without much information about the ramification index at each point.

We now determine a second class of groups which satisfy $N(k)$. Recall that if G is an arbitrary group and p is a prime dividing the order of G, then the subgroup of G generated by all of the Sylow p subgroups of G is either a characteristic proper subgroup of G or G itself.

Proposition 2.6. *Let $k > 1$ be an arbitrary positive integer and let H be a group whose order is divisible by a prime $p > k$. Suppose further that H has no characteristic subgroup which contains all the Sylow p subgroups of H. Then H satisfies $N(k)$.*

Proof. Assume $H \le G$ with $[G : H] = k' \le k$. Let P be a Sylow p subgroup of H and let n denote the number of Sylow p subgroups of H. Since $p > k$, P is a Sylow p subgroup of G.

We have the following:

$$[G : N_G(P)][N_G(P) : N_H(P)] = [G : H][H : N_H(P)]. \tag{5}$$

The first factor is the number of Sylow p subgroups of G, and must be of the form $n + t$ for some nonnegative integer t. In addition, p divides t, since $n \equiv 1 \bmod (p)$ and $n + t \equiv 1 \bmod (p)$. Thus, we have

$$(n+t)[N_G(P) : N_H(P)] = k'n. \tag{6}$$

From (6) it is obvious that $[N_G(P) : N_H(P)] \le k' < p$. However, from Eq. (6) we obtain $[N_G(P) : N_H(P)] \equiv k' \bmod p$, which implies $[N_G(P) : N_H(P)] = k'$. Thus, $t = 0$, so there are n Sylow p subgroups of G and they are all contained in H. Thus, H contains the characteristic (in both H and G) subgroup which they generate. By assumption they generate H, thus H is a normal, in fact, characteristic subgroup of G. □

Corollary 2.7. *A simple group whose order is divisible by a prime $p > k$ satisfies $N(k)$.*

Theorem 2.8. *Let X be a Riemann surface of genus greater than one, let H be a nonabelian simple group and suppose $H \le Aut(X)$. Let p be the largest prime dividing* *the order of H, let $Y = X/H$, and let $g \geq 1$ denote the genus of Y. Assume n points of Y are ramified in X and the ramification indices at these points are $e_1, \ldots, e_n$. Let k be computed, as in the Remark, for the sequence $(g; e_1, \ldots, e_n)$. If $H \neq Aut(X)$, then one of the following occurs:*

1. *H is a maximal subgroup of a simple group G and $p \leq [G : H] \leq k$.*
2. *$H \leq Aut(N)$ where N is a characteristically simple, nonsolvable group with $p \leq |N| \leq k$.*
3. *$H \leq GL(n,q)$ where $p \leq q^n \leq k$.*
4. *A nonidentity element of $Aut(Y)$ lifts to $Aut(X)$.*

If $p > k$ then 4 occurs.

Proof. Let $G \leq Aut(X)$ be minimal with respect to containing H, thus $H < G \leq Aut(X)$ and H is maximal in G. If $H \lhd G$ then a nonidentity automorphism of X/H lifts to X, which yields case 4. Assume now that cases 4 and 1 do not occur. Then $H < G$ and G is not simple, thus there exists $N \lhd G$. Since H is simple, $N \cap H = id$ and by minimality $G = HN$. Also by minimality, N must be characteristically simple. If N is solvable, this yields that N is elementary abelian, thus $|N| = q^n$ and H embeds in $Aut(N) \cong GL(n,q)$ which yields case 3. If N is not solvable, then H embeds into the automorphism group of the nonsolvable, characteristically simple group N. This yields case 2. Corollary 2.7 yields the bounds concerning p. This completes the proof. $\square$

Most simple groups are known to be generated by two elements and it is conjectured that all finite simple groups can be thus generated. Let Y be a Riemann surface of genus $g \geq 3$ with a trivial automorphism group and let Λ be a Fuchsian group of signature $(g; 0)$ such that U/Λ is biholomorphic to Y. Let H be a nonabelian simple group generated by g or fewer elements, and suppose the order of H is divisible by a prime greater than $84(g-1)$. Then certainly H is a homomorphic image of Λ; let Γ denote the kernel of the homomorphism. Then $X = U/\Gamma$ admits H as a group of automorphisms and $Y = X/H$. Since Y has trivial automorphism group, we conclude from Theorem 2.8 that $H = Aut(X)$.

A further example pertaining to this theorem is given in Section 5.

3. Nilpotent automorphism groups

In this section we consider nilpotent fixed-point free automorphism groups. Throughout this section we assume g is an integer greater than one. We associate a set of powers of primes to g denoted by π_g. If p is a prime and $p^k \mid 2(g-1)$, then $p^k \in \pi_g$. If $k \geq 1$ and p^k does not divide $2(g-1)$, but p^{k-1} does, then $p^k \in \pi_g$ if and only if there is an integer solution for t in

$$\frac{2(g-1)}{p^k} \leq t \leq \frac{2(g-1)}{p^{k-1}(p-1)}. \tag{7}$$

Remark. One easily deduces that:

1. If p does not divide $2(g-1)$ and $p(p-1) \leq 2(g-1)$, then $p \in \pi_g$, since in this case the right-hand and left-hand sides of (7) will differ by at least 1.

2. If $p \geq g-1$, then $p \notin \pi_g$ unless $p = g, p = g-1$, or $p = 2g-1$. In these three cases $p \in \pi_g$.

3. If 2^k is the largest power of 2 which divides $2(g-1)$, then $2^r \in \pi_g$ for all $r \leq k+1$.

Therefore, in practice, one employs (7) only for odd primes $p < g-1$ which satisfy $p(p-1) > 2(g-1)$ but which do not divide $(g-1)$, and for odd prime powers p^k such that p^k does not divide $(g-1)$ but p^{k-1} does.

We define a set of primes Π_g. A prime q is in Π_g if and only if q divides $p-1$ for a prime in π_g or q divides $p^r - 1$ for a prime power p^r in π_g. We will refer to elements of Π_g as unfavorable primes (for g). Primes not in Π_g will be called favorable primes. It follows easily from the Remark that 3 and 4 are elements of π_g for all g, hence 2 and 3 are unfavorable primes for all g.

Let H be a nilpotent group. Let H_1 be the product of the Sylow q subgroups for all unfavorable primes q, and let H_2 be the product of the Sylow q subgroups for all the favorable primes q. Note that $H = H_1 + H_2$ and both H_1 and H_2 are normal subgroups of H.

Theorem 3.1. *Let X be a Riemann surface which admits a nilpotent, fixed point free group of automorphisms H. Assume the 2 Sylow subgroup of H has class two or less. Let $Y = X/H$, and let $g \geq 2$ denote the genus of Y. Let $H = H_1 + H_2$ be the decomposition of H as described above. Let Z denote the Riemann surface X/H_2. Since $H_2 \lhd H$, elements of H induce automorphisms of Z. Then $H = Aut(X)$ if and only if no element of $Aut(Z)$ not contained in H lifts to an automorphism of X.*

Before giving the proof we state an immediate corollary.

Corollary 3.2. *With notation as above, suppose the order of H is divisible only by favorable primes. Then $H = Aut(X)$ if and only if no nonidentity element of $Aut(Y)$ lifts to an automorphism of X. In particular, if the order of H is divisible only by favorable primes and if $Aut(Y)$ is trivial, then $H = Aut(X)$.*

Our proof depends on the following two theorems.

Theorem 3.3 (Deskins et al.). *Let H be an arbitrary nilpotent group and suppose the class of the 2 Sylow subgroup of H is two or less. Assume H is a maximal subgroup of a group G. Then G is solvable.*

Proof. See [5, p. 445]. □

Theorem 3.4 (Galois). *Let G be a solvable primitive permutation group, let N be a minimal normal subgroup of G, and let G_1 be an isotropy subgroup. Then N is regular, has prime power order, and is elementary abelian. In addition $G = NG_1$ and $G_1 \bigcap N = id$.*

Proof. See [5, p. 159]. □

Proof of Theorem 3.1. If an automorphism of Z not contained in H lifts to X, then obviously $H < Aut(X)$ with strict inequality holding. Conversely, suppose $H \neq Aut(X)$. We will show there is a nonidentity automorphism of Z which lifts to X. Let G be any subgroup of $Aut(X)$ which is minimal with respect to containing H. If $H \triangleleft G$, we are done, so we may assume H is maximal, but not normal in G.

Let $j = [G : H]$ and let G act on the left cosets of H by left multiplication. Thus, G maps onto a subgroup of the symmetric group on j letters S_j. Let ϕ denote this map, (thus $\phi(g)$ maps xH to gxH) and let K be the kernel of ϕ. Then $K = core_G(H)$, the core of H in G; it is the largest normal (in G) subgroup contained in H. Thus, $G' = G/K$ is a permutation group; since $H' = H/K$ is maximal, G' is a primitive permutation group on j letters. Theorem 3.3 yields that G' is solvable. Theorem 3.4 yields that there exists a unique minimal normal subgroup of G' which we denote by N', and that $N' \cap H' = id$, $G' = H'N'$, and N' is elementary abelian. Let $m = p^{\alpha} = |N'|$. We note that since $K = core_G(H)$, H' contains no normal subgroups except for the identity.

Lemma. *p does not divide the order of H'.*

Proof. Assume p does divide the order of H'. Since H' is nilpotent, let P be the unique normal Sylow p subgroup of H'. We note that $H' \leq N_{G'}(P)$ so by minimality either $N_{G'}(P) = H'$ or $N_{G'}(P) = G'$. But P must be contained in a Sylow p subgroup of G', say $P < P'$, with strict inequality holding. However, in p groups a subgroup is always strictly contained in its normalizer. Thus, there are elements of P', not in H' which normalize P. Thus, $N_{G'}(P) = G'$. But this contradicts that H' contains no normal subgroups. Thus, p does not divide the order of H'. □

We now translate this information into the language of Fuchsian groups. Recall that X is an unbranched cover of Y, thus X is an unbranched cover of $X' = X/K$ and X' is an unbranched cover of Y. Let $\Gamma \leq \Delta \leq \Lambda$ be Fuchsian groups such that Δ is fixed point free, $\Delta/\Gamma \cong H', \Lambda/\Gamma \cong G'$, $Y = U/\Delta, X/G = U/\Lambda$, and $X' = X/K = U/\Gamma$. Let Q be a point of U/Λ. Since U/Λ is the orbit space X/G, each point of X which lies over Q must have the same branch index. This forces each point of Y which lies over Q to have the same branching index. Since $|N'| = p^{\alpha}$, any ramification of Y over U/Λ must be a power of p. However, since N' is elementary abelian, there are no elements of G' which have order p^k with $k > 1$. Since $\Lambda/\Gamma \cong G'$, and since Γ contains no elliptic elements, the only elliptic elements in Λ have order p. Thus, Λ has signature $(g';0)$

or $(g'; p, p, p, \dots, p)$, where in the former case $g' \geq 2$ and in the latter case $g' \geq 0$. The Riemann Hurwitz formula (1) yields

$$2(g-1) = 2p^{\alpha}(g'-1) + rp^{\alpha-1}(p-1). \tag{8}$$

Lemma. *U/A does not have genus zero.*

Proof. Assume U/A has genus zero. Then A is generated by elements of order p. Thus, G' is generated by elements of order p. But this says that G' is generated by elements of N'. This contradicts that $N' \lhd G'$. □

Let q be any prime dividing the order of H', and let Q be the Sylow q subgroup of H'. Since H' is a maximal subgroup of G' and H' contains no normal subgroups, $H' = N_{G'}(Q)$. Thus, for each prime q dividing $|H'|$, there are $m = p^{\alpha}$ Sylow q subgroups of G'. In particular, q divides $p^{\alpha} - 1$ for each q which divides the order of H'. We will show that p^{α} is an element of π_g, and thus $q \in \Pi_g$; let us assume this fact for the moment. Thus, if q divides the order of H, and if $q \notin \Pi_g$ then q does not divide $p^{\alpha} - 1$ and consequently q does not divide the order of $H' = H/K$. Thus, the Sylow q subgroup of H is a characteristic subgroup of H contained in K. Thus H_2, which is the subgroup generated by the Sylow subgroups corresponding to all primes not in Π_g is a characteristic subgroup of K. Since $K \lhd G$, we obtain $H_2 \lhd G$. In particular, G induces automorphisms of $Z = X/H_2$. Thus, there is an element of $Aut(Z)$ not contained in H which lifts to an automorphism of X.

To complete the proof it is necessary to show that $p^{\alpha} \in \pi_g$. This is accomplished by examining (8). Recall $g' \geq 1$ in (8) since the orbit space under G' cannot have genus zero.

If $p = 2$, then (8) yields that $2^{\alpha-1} | 2(g-1)$, thus $2^{\alpha} \in \pi_g$. Now assume p is odd. If $p^{\alpha} | 2(g-1)$, then by definition $p^{\alpha} \in \pi_g$. Thus, we may assume that p^{α} does not divide $2(g-1)$ for the odd prime p. Again from (8) it is clear that $p^{\alpha-1}$ divides $2(g-1)$. Thus,

$$\frac{(g-1)}{p^{\alpha-1}} = p(g'-1) + r(p-1)/2. \tag{9}$$

Note also that

$$\frac{(g-1)}{p^{\alpha-1}} = \frac{p(g-1)}{p^{\alpha-1}} - \frac{2(g-1)}{p^{\alpha-1}}(p-1)/2. \tag{10}$$

From elementary number theory we conclude that there exists an integer t such that

$$0 \leq g' - 1 = \frac{(g-1)}{p^{\alpha-1}} - \frac{t(p-1)}{2} \tag{11}$$

and

$$0 \leq r = tp - \frac{2(g-1)}{p^{\alpha-1}}. \tag{12}$$

This yields

$$\frac{2(g-1)}{p^{\alpha}} \le t \le \frac{2(g-1)}{p^{\alpha-1}(p-1)}. \tag{13}$$

Thus $p^{\alpha} \in \pi_g$. This completes the proof of the theorem. □

We give one more corollary.

Corollary 3.5. *Let X be a Riemann surface and let $H \le Aut(X)$ be a nilpotent, fixed point free automorphism group of odd order. Let $Y = X/H$, let $g \ge 2$ denote the genus of Y, and let $g-1 = 2^k m$ where m is odd. Assume that $|H|$ is not divisible by a prime less than or equal to $(g-1)/2$. In addition assume $|H|$ is not divisible by any of the following primes q:*

1. *$q = g-1$ if both $g-1$ and $2g-1$ are prime.*
2. *Any prime divisor q of $2^s - 1$ where $1 \le s \le k+2$.*
3. *$q = (3^{j+1}-1)/2$ if $g-1 = 3^j$ or $2(3^j)$.*
4. *$q = (5^{j+1}-1)/4$ if $g-1 = 2(5^j)$.*

Then $H = Aut(X)$ if and only if no nonidentity automorphism of Y lifts to X.

Note that there are only 7 primes less than one billion of the form $q = (3^{j+1}-1)/2$ or $(5^{j+1}-1)/4$.

Proof. It is sufficient to show that the unfavorable primes for g which are greater than $(g-1)/2$ are among the primes listed in 1–4 above. For the rest of the proof we assume q is an odd prime, unfavorable for g, which satisfies $q > (g-1)/2$. Assume first that q divides $p-1$ for some prime $p \in \pi_g$. If p divides $2(g-1)$, then since q divides $p-1$, $q \le (g-1)/2$, contradicting our assumptions about q. If p does not divide $2(g-1)$, then the remark at the beginning of this section yields that $q \le (g-1)/2$ or $q = g-1$. This latter case can occur only if $2g-1$ is prime. Thus, q satisfies 1 above. Now assume that q divides $p^i - 1$ for some $p^i \in \pi_g$ with $i > 1$, but q does not divide $p-1$. If $p = 2$ then q satisfies 2 above, so we may assume p is odd. If p^i divides $2(g-1)$, this again clearly contradicts that $q > (g-1)/2$. Since $p^i \in \pi_g$, we know p^{i-1} divides $g-1$, thus $p^i s = p(g-1)$ for some positive integer s. Thus,

$$q \,\Big|\, p^i - 1 < \frac{p(g-1)}{s}. \tag{14}$$

If $d \mid i$ for some integer d between 1 and i, then

$$q < \frac{p(g-1)}{s(p^d-1)} \tag{15}$$

contradicting that $q > (g-1)/2$. Thus i is prime. Thus

$$q \,\Big|\, \frac{p^i-1}{p-1} = (1 + p + \cdots + p^{i-1}) < \frac{p(g-1)}{s(p-1)}. \tag{16}$$

Since $q > (g-1)/2$, we obtain $s \leq 2$. Recall from the proof of Theorem 3.1 that since $p^i \in \pi_g$, p^i is the order of N', and the Riemann Hurwitz formula (8) holds for p^i. Thus, (8) yields

$$2p^{i-1}s = 2p^i(g'-1) + rp^{i-1}(p-1). \tag{17}$$

Recall in (17) that $g' \geq 1$. Thus, $s = 1$ and $p = 3$, or $s = 2$ and $p = 3$ or 5.

Assume now that $p = 3$. If $i = 2$, then (16) yields that $q = 2$, contradicting that q is odd. If i is an odd prime and $(3^i - 1)/2 = 1+3+\cdots+3^{i-1}$ is not prime, then $(3^i-1)/2$ is not divisible by a prime less than 5. In this case (16) yields that $q \leq (g-1)/2$, again contradicting our assumptions on q. Thus, $q = (3^i - 1)/2$ is prime. This yields 3 above. An analogous argument for $p = 5$ yields 4 above. □

4. Explicit examples with defining equations

In [7, 9], the defining equations of families of Riemann surfaces whose members have a trivial automorphism group are given. In [7] the following family is presented. Let $n > m+1 > 3$, and assume $n-1$ and m are relatively prime. Let C be the locus of the equation

$$f(x,y) = y^n + yp(x) + a_0 x = 0 \tag{18}$$

in $\mathbb{C}^2$. In (18), we assume $p(x) \in \mathbb{C}[x]$ has degree m and x but not x^2 divides $p(x)$. In addition, the coefficient of x^{m-1} in $p(x)$ is 0. The only restriction on a_0 is that it makes the locus of C nonsingular in $\mathbb{C}^2$.

Let C' be a nonsingular model for C. In [7] it is shown that C' may be considered as C with two additional points P and Q adjoined. The genus of C' is $n(m-1)/2$ and C' has a trivial automorphism group. Let R denote the point $(0,0)$. The functions x and y have the following divisors:

$$(x) = nR + (1-n)P - Q, \qquad (y) = R - mP + (m-1)Q. \tag{19}$$

We construct the following examples based upon (18).

Example 1. Let $f(x,y)$ be as in (18), let $f_1, \ldots, f_s$ be elements of $\mathbb{C}(x,y)$ and for each i with $1 \leq i \leq s$, let n_i be an upper bound for the number of points of C' where f_i has either a zero or a pole. Let $N = n_1 + \ldots + n_r$. For each i, let q_i be a prime satisfying $q_i > 6(2g + N - 1)$, let $q_1, \ldots, q_r$ be distinct, and assume f_i is not a q_ith power of an element of $\mathbb{C}(x,y)$. This latter condition can be achieved, for example, by choosing q_i greater than the degree of the pole divisor of f_i. For each i with $1 \leq i \leq r$, let α_i be an arbitrary positive integer. Define $Q_i = q_i^{\alpha_i}$. Consider the polynomial $F(T_1, \ldots, T_r) = \prod_{i=1}^{s}(T_i^{Q_i} - f_i)$. Let X be a nonsingular model for the function field defined by $f(x,y) = 0$ and $F(T_1, \ldots T_r) = 0$. Since for each i, f_i is not a q_ith power of an element of $\mathbb{C}(x,y)$, we obtain, $[\mathbb{C}(X) : \mathbb{C}(x,y)] = Q_1 Q_2 \ldots Q_r$. The Riemann surface X admits an automorphism group $H \cong \mathbb{Z}_{Q_1} + \cdots + \mathbb{Z}_{Q_r} \cong \mathbb{Z}_{Q_1 \ldots Q_r}$ and

$C' = X/H$. The points of C' which are ramified in X are the points where one of the f_i's has a zero or pole. Thus, for each q_i there is an element of H of order q_i which is not fixed point free on X. Since C' has a trivial automorphism group, Corollary 2.5 yields that $H = Aut(X)$.

Example 2. We now show how knowledge of an automorphism group with a genus zero quotient can be used to construct nontrivial automorphism groups with a genus g quotient for $g \geq 2$.

Let q and p be primes and let $q \equiv 1 \bmod p$. Let the positive integer j be a primitive pth root of unity in $\mathbb{Z}_q$, and let $d = (j^p - 1)/q$. In addition, let ε and λ denote primitive pth and qth roots of unity in $\mathbb{C}$, respectively. Consider the field extension of $\mathbb{C}(t)$ given by

$$z^q - \prod_{i=0}^{p-1} \left(\frac{t^p - 1}{(\varepsilon^{p-i}t - 1)^p} \right)^{j^i}. \tag{20}$$

Let Z be a nonsingular model for the function field $\mathbb{C}(t,z)$. It is easy to see that Z admits a nonabelian automorphism group H of order qp. This group has generators σ and τ where

$$\tau(t) = \varepsilon t, \qquad \tau(z) = \frac{z^j(\varepsilon t - 1)^{dp}}{(t^p - 1)^d}, \tag{21}$$

$$\sigma(t) = t, \qquad \sigma(z) = \lambda z. \tag{22}$$

The function field corresponding to the Z/H is $\mathbb{C}(t^p)$, thus Z/H is the Riemann sphere. There are exactly three points of the Riemann sphere which are ramified in Z; the ramification index of the points $0, \infty$, and 1 are p, p and q, respectively.

We use the above automorphism group to construct a Riemann surface X which has a full automorphism group of order pq. Let $f(x,y)$ and $g(t,z)$ be as in (18) and (20), respectively, and let X be a nonsingular model for the function field defined by $f(x,y) = 0$, $g(t,z) = 0$, and $t^p - x = 0$. From (19) we see $t \notin \mathbb{C}(x,y)$. It is clear that X admits a group of order pq generated by σ and τ where σ and τ are defined as in (21) and (22) and act as the identity on y. We call this automorphism group H again and note that $C' = X/H$. In addition, X is unbranched over C' at points where $x \neq 0, 1$, or ∞. Note that R, P, and Q are the only points where x takes on the value 0 or ∞. Considering (18), there are at most n points of C' at which x takes on the value 1. Thus, there are a maximum number of $n+3$ points of C' which are ramified in X. Calculating k as in the Remark in Section 2 we obtain $L = pq$ and $k = 6pq(2g + n - 2)/(pq - 6)$. Thus, if $p \geq k$ then $H = Aut(Y)$. A short calculation shows that $p \geq k$ if $p > 6(nm - 2)$.

Example 3. The construction of fixed point free examples is more delicate. For example, if a cyclic automorphism group is defined by the equation $T^q - g = 0$, where q is prime and $g \in \mathbb{C}(C')$, then every pole or zero of g must be a multiple of q. Functions which satisfy such a constraint on their divisors may be difficult to construct. However, we offer the following example.

Let C' be as above. In addition, assume m and n are chosen so that $p = 2g + 1 = n(m-1)+1$ is prime. Let $f(x, y)$ be as in (18) and let the Riemann surface X be a nonsingular model for the curve defined by $f(x, y) = 0$, and $w^p - yx^{m-1} = 0$. Then $Aut(X)$ is cyclic of prime order and C' is the orbit space of X under $Aut(X)$.

Proof. Note from (19) that $(yx^{m-1}) = pR - pP$. It is obvious that X admits an automorphism σ of order p which fixes x and y and maps w to λw where λ is a pth root of unity. Let H denote the group generated by σ; thus $X/H = C'$. It is easily shown that H is fixed point free. A short calculation shows that p is not one of the unfavorable primes listed in Corollary 3.5. Since C' admits no nontrivial automorphisms, the corollary yields that $H = Aut(X)$. $\square$

5. Simple fixed point free groups with a genus three orbit space

If a particular genus is chosen to be the orbit space of a Riemann surface under a group of automorphisms, the Riemann Hurwitz formula reveals more information than in the general case. As an example, in this section we restrict our attention to Riemann surfaces X which admit a simple fixed point free automorphism group H such that X/H has genus three. Throughout this section X will denote a Riemann surface of genus $g_x, H \leq Aut(X)$ is a simple fixed point free automorphism group, and X/H has genus 3. Proposition 1.2 yields that $|H| = (g_x - 1)/2$. Thus, if $H \neq Aut(X)$, then for any group G with $H < G \leq Aut(X)$, we have $|G|$ is a multiple of $(g_x - 1)/2$. Using the Riemann Hurwitz formula (1), it is not difficult to determine the restrictions it places on the order of $|G|$. Specifically, $[G : H] \in \Omega$, where

$$\Omega = \{2, 3, \ldots, 26, 27, 28, 30, 32, 33, 36, 40, 42, 48, 60, 72, 80, 96, 168\}. \tag{23}$$

Theorem 5.1. *Let X be a Riemann surface, let $H \leq Aut(X)$ be a nonabelian simple fixed point free automorphism group and let X/H have genus 3. Assume H is not $PSL(3,3)$ or $PSL(5,2)$ and let p be the largest prime dividing the order of H. Assume H is not a maximal subgroup of a simple group G with $p \leq [G : H] \in \Omega$. Then $H = Aut(X)$ if and only if there is no nonidentity automorphism of X/H which lifts to X.*

Proof. This follows directly from Theorem 2.8. The only characteristically simple non-solvable groups with order in Ω are A_5 and $PSL(2,7)$. The only elementary abelian groups of order $p^k \in \Omega$ are of order $3^2, 3^3, 5^2$ or 2^k with $k = 1, \ldots, 5$. The corresponding $GL(k, p)$ groups contain the following simple groups [1]: $PSL(4,2), PSL(5,2), PSL(3,3)$, $PSL(2,7), A_5, A_6, A_7, A_8$. All of these groups are contained as a maximal subgroup of a simple group, with index in Ω, except for $PSL(5,2)$, and $PSL(3,3)$. Thus Theorem 2.8 yields the conclusion. □

Using Theorem 5.1 and Theorem 2.4 is not difficult to prove the following.

Proposition 5.2. *Let $H = PSL(2,q^n)$ where $q^n > 11$. Let X be a Riemann surface, let $H \leq Aut(X)$ be fixed point free, and let X/H have genus 3. Then $H = Aut(X)$ if and only if no nonidentity automorphism of X/H lifts to X.*

Proof. If $q^n > 11$, then $PSL(2,q^n)$ contains no subgroup of index less than q^n [5]. Thus if $q^n \geq 168$, Theorem 2.4 yields the conclusion. If $q^n < 168$, one can apply Proposition 2.3 or one can use [1], to check that $PSL(2,q^n)$ is not contained as a maximal subgroup of a simple group with index in Ω. □

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